Chapter III

Transition to Electrostatics

Electromagnetic Fields and Waves

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Introduction:

In the previous chapter, we studied the performance of Transmission Lines as circuit elements using an RLCG model. In other words, we used the circuit elements model to study the wave propagation properties of the transmission media. This RLCG model is in fact an approximation and oversimplification of the physical TL configuration. The only way to appreciate this fact is for us to examine how such a model is developed. This will be the topic of the following chapters of this book where we will be studying relevant electromagnetic topics leading to the tools that allow us to determine these "Circuits" model parameters. In the process, we will arrive at more powerful static and dynamic electromagnetic analysis tools. These tools will enable us to study phenomena that are more complex without resorting to the over-simplification of using circuit element modeling.

You see, the issue with the use of approximate models is that you typically cannot predict when the use of these models will fail in yielding reliable results. In the early discussion we had in Chapter I, we showed the need to resort to TL analysis techniques when the size of the components is no longer negligible compared to the wavelength. One can predict as a result that "approximate" circuit model would fail as the frequency goes high. Considering the cases of mm waves and optical waves, it is not practical to represent such cases using Circuits models at all. The only way to deal with these cases is through electromagnetic analysis.

There are other classes of problems that require a decent appreciation of electromagnetic analysis tools. Examples include (but not limited to):

- Antenna analysis and design
- Wireless communication systems and devices
- Semiconductor devices and integrated circuits
- Computer chips (processors, memory, etc.)
- Circuit boards (planar, multilayer, flexible, etc.)
- Cables, connectors, and adaptors.
- Discrete components (R, C, L, transformers, etc.)
- Power systems and machines (transformers, motors, generators, etc.)
- Magnetic devices (sensors, relays, actuators, etc.)
- MEMS and MW devices
- RADAR and remote sensing
- Microwave heating and curing
- EMI/EMC analysis
- Nanotechnology
- Photonics

To appreciate the argument we are making here, let us discuss an example of capacitance; one of the fundamental circuit elements that you have learned about before and will be discussed in more detail in later chapters.

A capacitor is simply formed of two conductors isolated from each other by a dielectric (insulator). Physical capacitors that we often see in circuit boards and electrical and electronic devices take many shapes; typical shapes are depicted in Figure 3.1. The figure shows the physical shapes of some "Ceramic Disc" capacitors and next to them is a simplified sketch of their internal physical structure.

Figure 3.1

The typical circuit analysis for such a device is a capacitive circuit element in the nominal capacitance value of the capacitor. Careful analysis shows that such simplified model is not accurate enough for certain applications, especially at high (RF) frequencies. Other "packaging" and "parasitic" elements are present that must be accounted for. Examples are dielectric leakage conductance (G), packaging capacitance (C_p), lead inductance (L_p, due to induced magnetic fields), and lead (wire) resistance (R_p). These are shown in Figure 3.2.

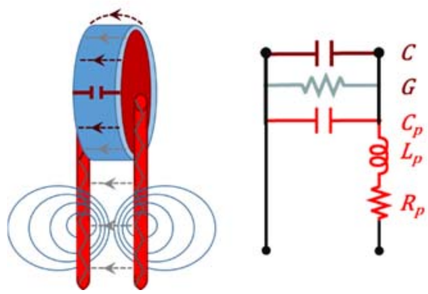


Figure 3.2

That being said, to be able to develop the details of such model, we need to understand and relate to the physical phenomenon of resistance, inductance, capacitance, and how the geometrical configuration and material properties among other factors influence their behavior. This is exactly what we are going to "introduce" the learner to in this text book. We use the term "introduce" since the topics involved are far wide and cannot be put all in one book, especially at the introductory level to the subject of EM.

More about the RLCG Parameters

As we recall from past studies of circuit theory, resistance and conductance, are both two sides to the same phenomenon. Both have to do with the ability of the charges to move and the "resistivity" or "conductivity" of the media they move or "flow" in. The conductance is simply the mathematical inverse of the resistance. In this regard, we learned that materials are divided into two main categories: conductors and insulators. A third category was added later in our studies as the "semiconductors" which is by default an insulator that conducts in certain ways under certain conditions. Conductors are those materials that have amble of mobile "free" charges that can move if forced to. The motion of the free charges is known as the electric current. The amount of "impedance" this flow encounters is called the resistance, which depends on the properties of the material and the geometrical configuration and dimensions. The current flow against the resistance causes an electric potential drop and hence power dissipation.

As for the capacitance, it is associated with the presence of an insulating material (dielectric) that isolates two conductor surfaces. The capacitor can be charged (and discharged) by altering the charge amount on the conductor terminals of the capacitor. This, in turn, results in a potential difference across the capacitor terminals which is an indication of energy storage in the capacitor. Energy can be stored (and retrieved) in the capacitor without loss as long as the insulating material is lossless and the two conductors are perfect. An ideal insulating material does not allow direct current flow in a capacitive circuit element. Yet in a dynamic situation, and as a result of charge fluctuations, a dynamic current is modeled in the circuit. We will learn later about this current as the "displacement" current in the dielectric insulator.

The inductance has to do with the magnetic behavior of the media as seen by an electric circuit. Typically, a conductor interaction (linkage) with a medium (magnetic or nonmagnetic) represents an inductance as modeled in the electrical circuit containing that conductor. A time varying current flowing in the inductor results in a potential difference across its terminals while a direct current does not produce any. Both direct and time varying currents result in static and dynamic energy storage in the inductance with no associated dissipation or loss. The inductance value depends on the magnetic (or non-magnetic) properties of the medium, its geometrical configuration and dimensions, as well as the geometrical configuration and dimensions of the conductor (coil) and its shape of interaction (linkage) with the medium.

With all this background information about the Capacitance. Resistance, and Inductance, it is clear that we need to study the electric and magnetic phenomena associated with static and dynamic charges and their interactions with dielectrics, conductors, and magnetic/non-magnetic materials. We will do so in the next few chapters starting with static charges, charges moving with constant velocity generating direct currents, and finally the general case of time varying currents. The first two steps of this study will fall under the "statics" category, while the third one will be under "dynamics".

Before we indulge ourselves in this Static/Dynamic study, we will need to review some mathematical tools that we cannot do without. This review is covered in the Addenda of this chapter, while we start our Statics study in the next chapter, Chapter IV.

Addendum I

Coordinate Systems

Introduction

Coordinate systems are used to define positions/locations in space relative to some arbitrary origin. Three orthonormal coordinates sufficiently define points and locations in three dimensional space. There are three acknowledged coordinate systems in this regard which we will review here in this chapter; they are: the Cartesian, the Cylindrical, and the Spherical coordinate systems.

The Cartesian coordinate system:

The Cartesian coordinate system is the most commonly used as it utilizes three orthonormal constant-direction coordinates. Also, because most of the physical structures are based on rectangular shapes. The convenience of having fixed direction for its three coordinated pays off in many analyses as we will see in our studies in this book and elsewhere.

In Cartesian coordinates, we establish three orthonormal lines/directions/coordinates that stem from an arbitrary origin. Typically, two of these coordinates are laid flat on a horizontal plane; these are typically chosen as the x and y coordinates (see Figure 3.3). A third coordinate is made to emerge orthonormally to the x-y plane and is called the z-coordinate. The positive directions of the three coordinates are chosen to form a right-hand system in the x-y-z-x order (see Figure 3.4). The figure also shows the three unit vectors $\vec{a}_x, \vec{a}_y, and\ \vec{a}_z$ for this coordinate system.

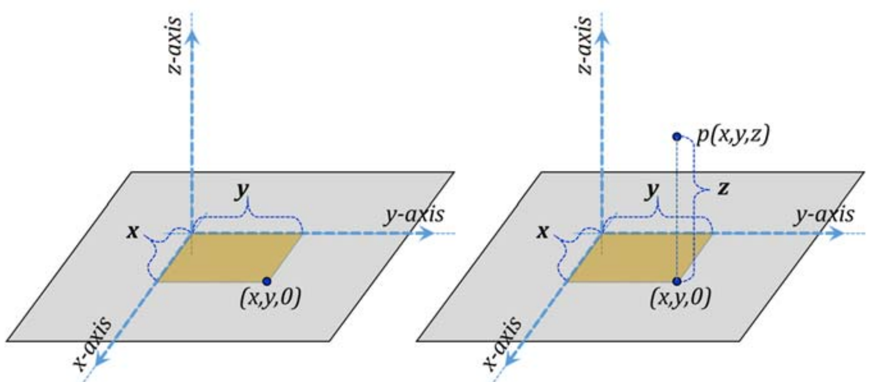


Figure 3.3

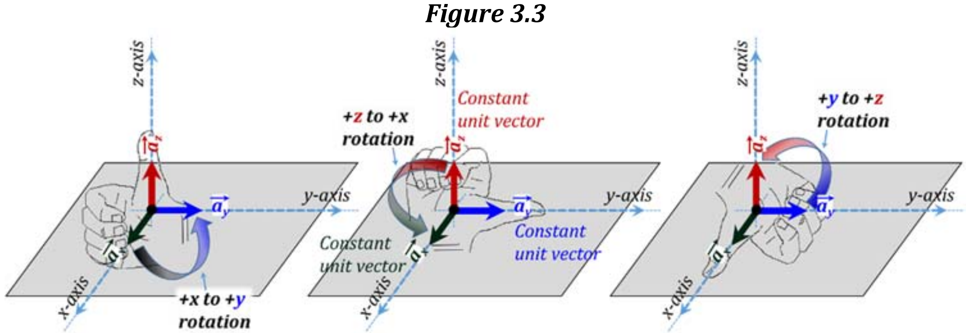


Figure 3.4

In the Cartesian coordinate system, a point p is specified by the three coordinates x, y, and z; these are the normal distances between the point p and the y-z plane, z-x plane, and the x-y plane, respectively (Figure 3.5).

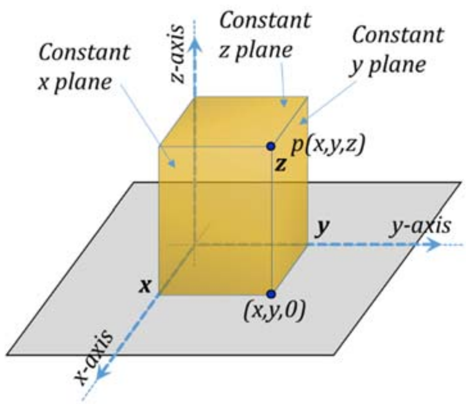


Figure 3.5

For integration purposes, incremental elements at p correspond to increasing the coordinate values x, y, and z by Δx, Δy, and Δz, respectively, Figure 3.6. The corresponding volume for the incremental rectangular prism is the product of all three sides Δv=Δx Δy Δz.

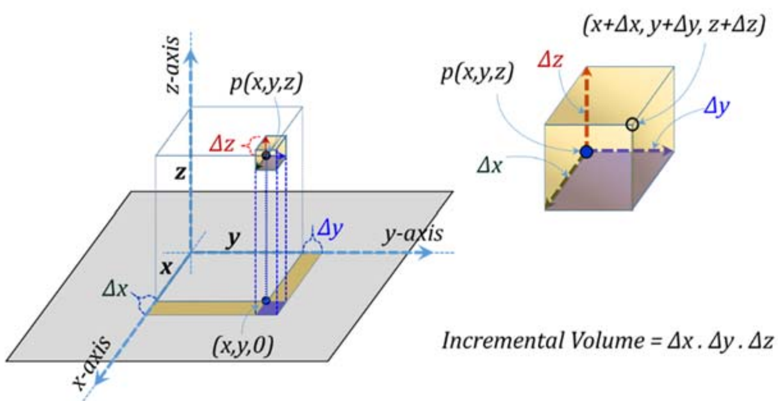


Figure 3.6

The vector representation of the three incremental lengths is given in Figure 3.7 while the corresponding three incremental surface area vectors are shown in Figure 3.8. More discussion of vectors will follow in the next Addendum of this chapter.

We need to recognize that the unit vectors $\vec{a}_x, \vec{a}_y, and\ \vec{a}_z$ are constant for all points in space for this coordinate system. They are constant in magnitude (which is unity) and constant in directions as well. Length vectors have directions along the lengths themselves, while the direction of an area vector is in the outward normal of the face of that area.

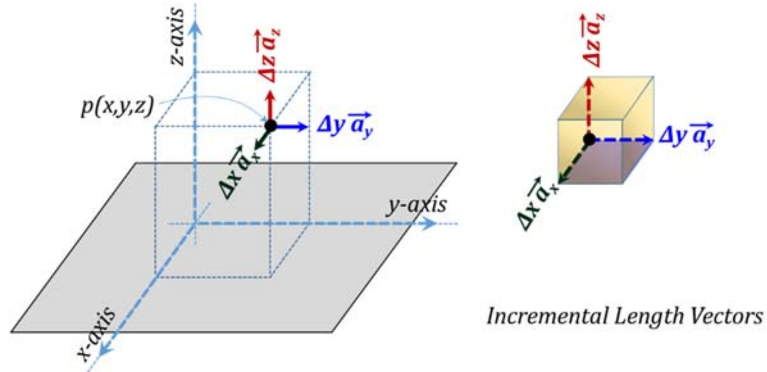


Figure 3.7

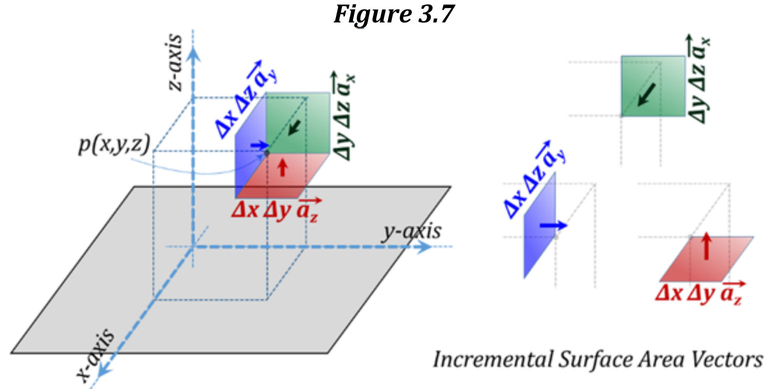


Figure 3.8

The Cylindrical coordinate system:

Cylindrical coordinates are useful to express positions and location when a spatial configuration is essentially cylindrical. In this case, Figure 3.9 shows the three orthonormal Cylindrical coordinates ρ, φ, and z as laid over the corresponding Cartesian system chosen with a common z-axis. In this figure, ρ is the radius of a vertical cylinder passing by the position, φ is the angle between the vertical plane containing the z-axis and the position and the z-x plane; finally, z is the height of the position above the x-y plane. The unit vectors along the three coordinates are demonstrated in Figure 3.10. The positive directions of the three coordinates is chosen to form a right-handed system in the ρ-φ-z-ρ order, Figure 3.11.

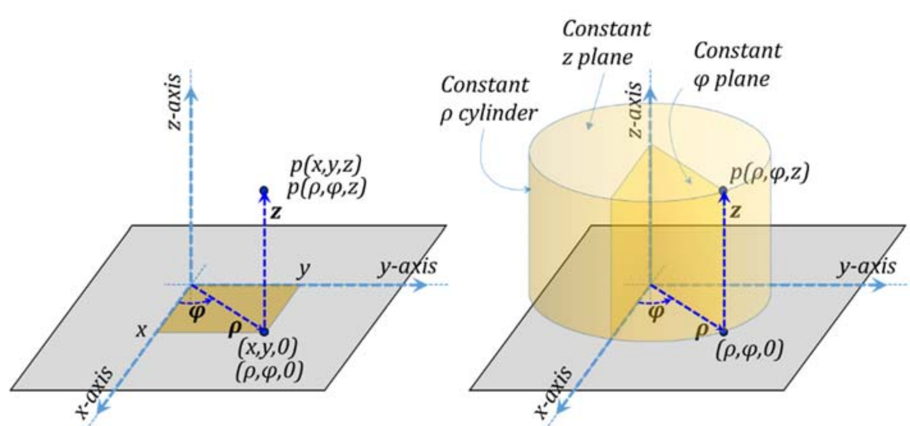


Figure 3.9

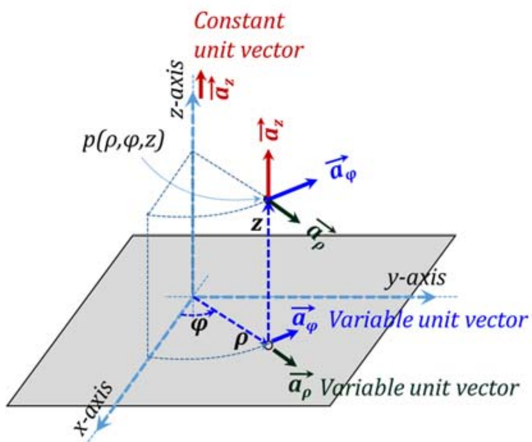


Figure 3.10

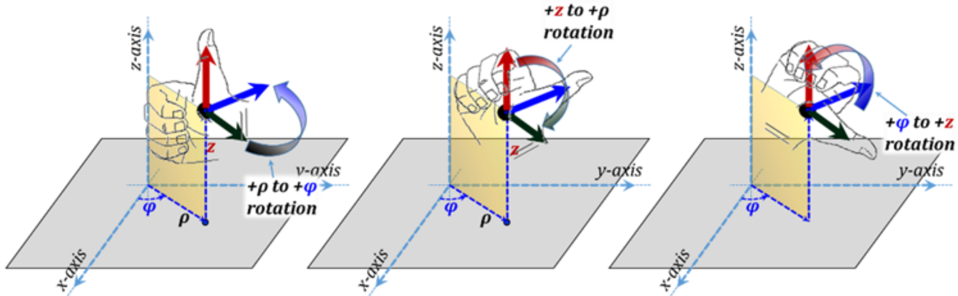


Figure 3.11

For integration purposes, incremental elements at p correspond to increasing the coordinate values ρ, φ, and z by Δρ, Δφ, and Δz, respectively, Figure 3.12. The side lengths of the corresponding incremental "rectangular" prism are Δρ, ρΔφ, and Δz, respectively. The corresponding incremental volume is the product of all three sides Δv= Δρ ρΔφ Δz.

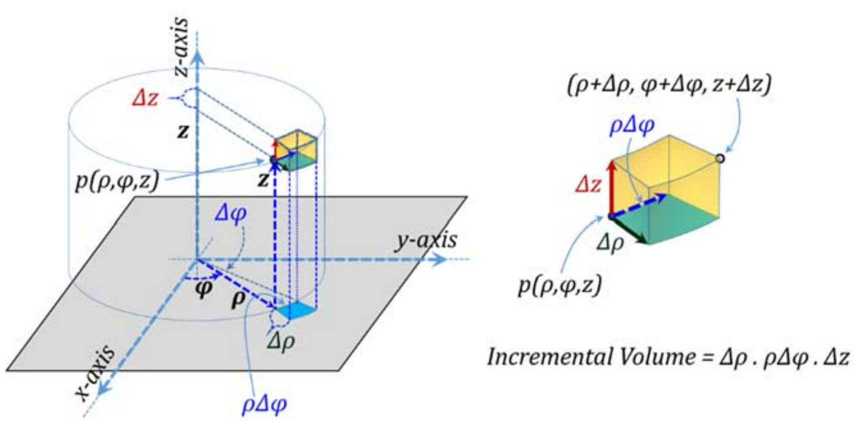


Figure 3.12

The vector representation of the three incremental lengths is given in Figure 3.13 while the corresponding three incremental surface area vectors are shown in Figure 3.14. Unlike the case with Cartesian coordinates, the unit vectors $\vec{a}_\rho$ *and* $\vec{a}_\varphi$ are variable vectors as they both vary in direction as we move around in space. However, the unit vector $\vec{a}_z$ is the same as that of Cartesian coordinates and hence is a constant vector for all points in space.

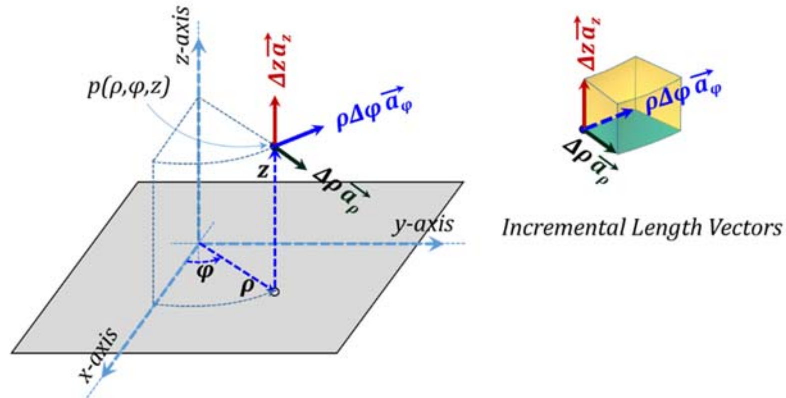


Figure 3.13

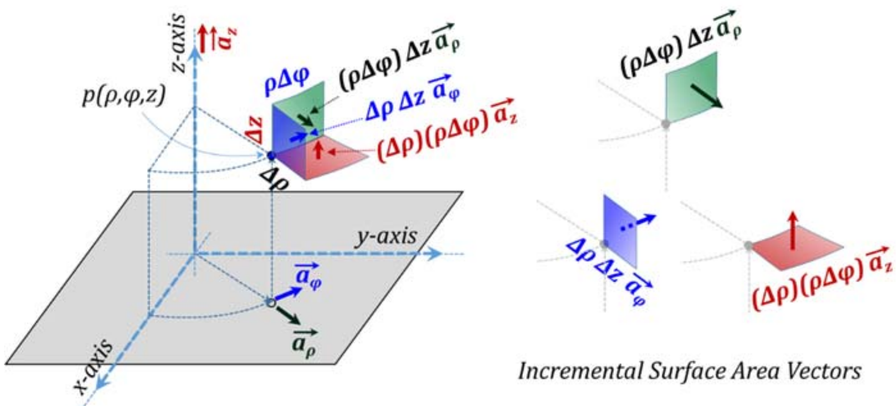


Figure 3.14

The Spherical coordinate system:

Spherical coordinates are typically used in cases where a spatial configuration is essentially spherical. The three orthonormal Spherical coordinates are r, θ, and φ, see Figure 3.15. This figure shows all three coordinate systems simultaneously in order to demonstrate their relationship to each other.

As shown in the figure, r is the radius of a sphere passing by the position p while being concentric with the origin. The angle θ is the second coordinate parameter and is the angle between the radial line connecting the position to the origin and the z-axis. The angle θ can also be defined as the angle of the cone whose head is the origin, coaxial with the z-axis and passing by the position p. Finally, the angle φ is the same as defined for Cylindrical coordinates, i.e., the angle between the vertical plane containing the z-axis and the position p and the z-x plane.

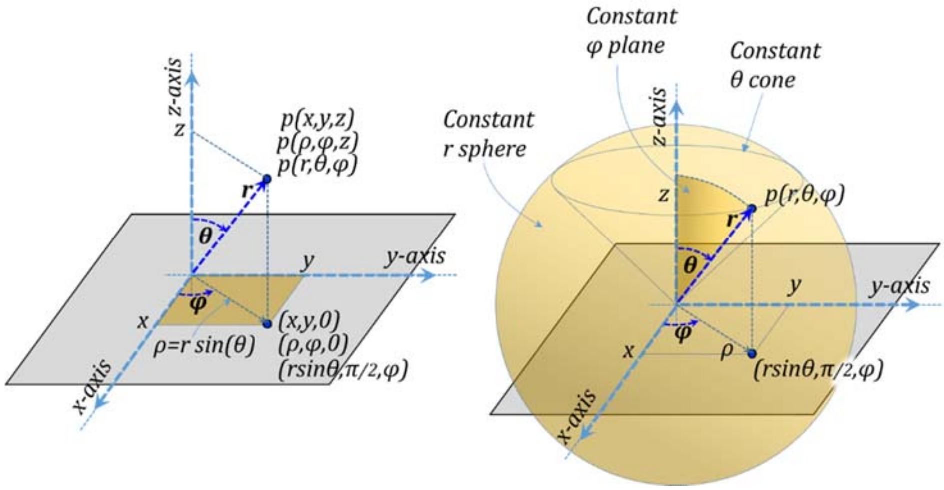


Figure 3.15

In Figure 3.16, we demonstrate the three orthonormal unit vectors $\vec{a}_r$, $\vec{a}_\theta$, and $\vec{a}_\varphi$. Again, the positive directions of the three coordinates is chosen to form a right-handed system in the r-θ-φ-r order, Figure 3.17.

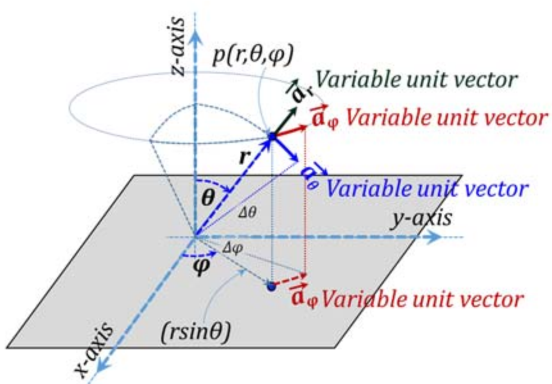


Figure 3.16

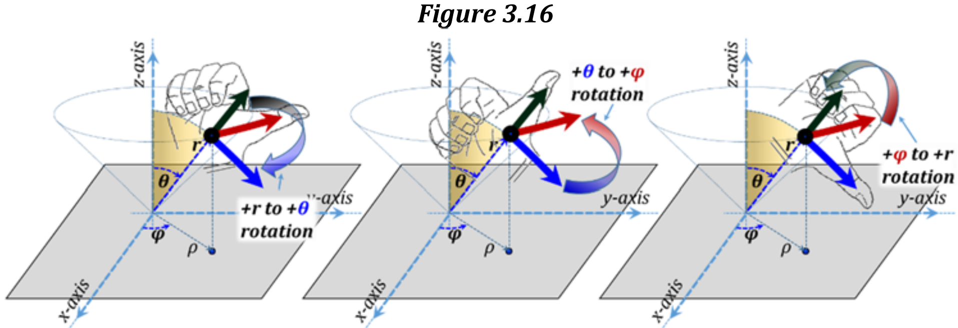


Figure 3.17

For integration purposes, incremental elements at p correspond to increasing the coordinate values r, θ, and φ by Δr, Δθ, and Δφ, respectively, Figure 3.18. The side lengths of the corresponding incremental "rectangular" prism are Δr, rΔθ, and rsin(θ)Δφ, respectively. The corresponding incremental volume is the product of all three sides Δv= Δr rΔθ rsin(θ)Δφ.

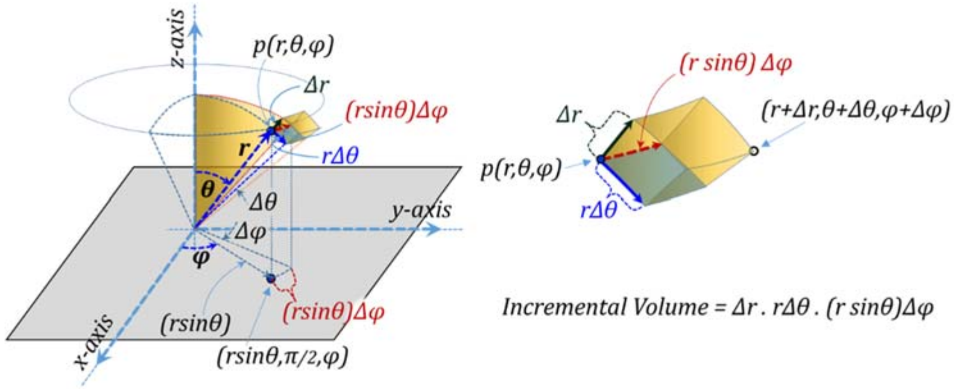


Figure 3.18

The vector representation of the three incremental lengths is given in Figure 3.19 while the corresponding three incremental surface area vectors are shown in Figure 3.20. Needless to say, all three unit vectors $\vec{a}_r, \vec{a}_\theta, and\ \vec{a}_\varphi$ are variable vectors as they all vary in direction as we move around in space.

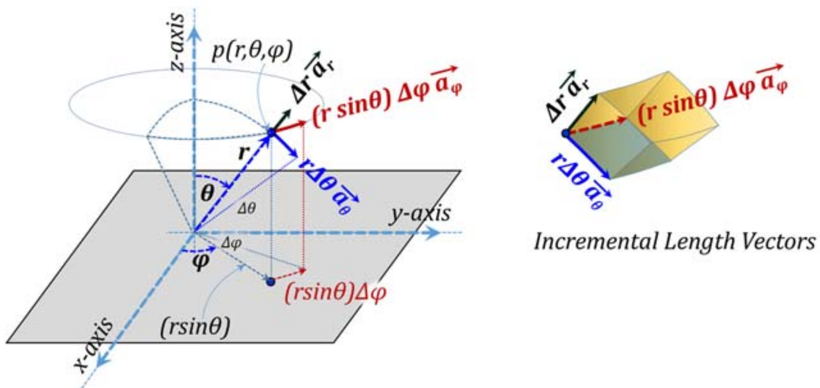


Figure 3.19

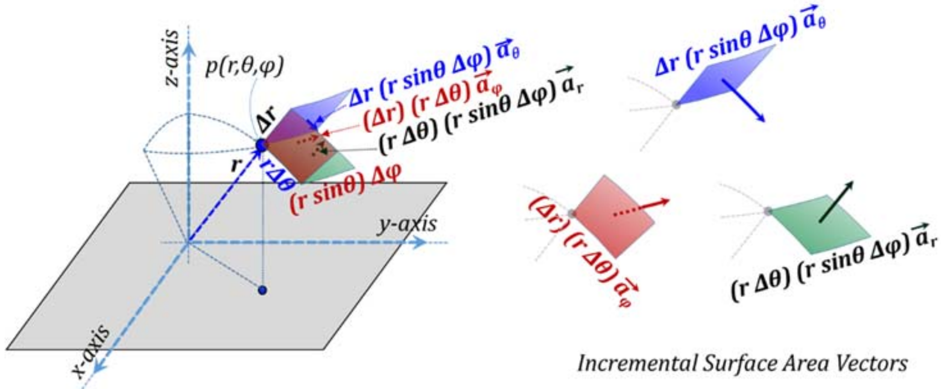


Figure 3.20

Relationships between Coordinate Systems:

Frequently, in the course of analytical procedures, we would be faced with the need to convert from one coordinate system to another. Typically, the conversion would be needed between Cylindrical and Cartesian and between Spherical and Cartesian. In some cases this is needed for convenience in carrying out a "difficult" integration or to identify with a specific coordinate-related phenomenon. However, in many cases it is done to avoid dealing with variable unit vectors. Converting to Cartesian coordinates replaces the variable unit vectors with constant ones that facilitates integration procedures involving vectors.

Examining Figures 3.9 and 3.10 for Cylindrical coordinates and 3.14 and 3.15 for Spherical coordinates, we can write down the relationships for the coordinate parameters and the corresponding unit vectors as laid down in the tables below.

Table 3.1 - Coordinate Relations

Cylindrical – Cartesian	*Spherical – Cartesian*	*Cylindrical - Spherical*
$\rho = \sqrt{x^2 + y^2}$ $\varphi = tan^{-1}(y/x)$ $z = z$	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = cos^{-1}(z/r)$ $\varphi = tan^{-1}(y/x)$	$\rho = r\ sin\ \theta$ $\varphi = \varphi$ $z = r\ cos\ \theta$
$x = \rho\ cos\ \varphi$ $y = \rho\ sin\ \varphi$ $z = z$	$x = r\ sin\ \theta\ cos\ \varphi$ $y = r\ sin\ \theta\ sin\ \varphi$ $z = r\ cos\ \theta$	$r = \sqrt{\rho^2 + z^2}$ $\theta = cos^{-1}(z/r)$ $\varphi = \varphi$

Table 3.2 - Unit vectors Relations

Cylindrical – Cartesian	*Spherical – Cartesian*	*Cylindrical - Spherical*
$\overrightarrow{a_\rho} = cos\ \varphi\ \overrightarrow{a_x} + sin\ \varphi\ \overrightarrow{a_y}$ $\overrightarrow{a_\varphi} = -sin\ \varphi\ \overrightarrow{a_x} + cos\ \varphi\ \overrightarrow{a_y}$ $\overrightarrow{a_z} = \overrightarrow{a_z}$	$\overrightarrow{a_r} = sin\ \theta\ cos\ \varphi\ \overrightarrow{a_x} + sin\ \theta\ sin\ \varphi\ \overrightarrow{a_y} + cos\ \theta\ \overrightarrow{a_z}$ $\overrightarrow{a_\theta} = cos\ \theta\ cos\ \varphi\ \overrightarrow{a_x} + cos\ \theta\ sin\ \varphi\ \overrightarrow{a_y} - sin\ \theta\ \overrightarrow{a_z}$ $\overrightarrow{a_\varphi} = -sin\ \varphi\ \overrightarrow{a_x} + cos\ \varphi\ \overrightarrow{a_y}$	$\vec{a}_r = \vec{a}_\rho\ sin\ \theta + \vec{a}_z\ cos\ \theta$ $\vec{a}_\theta = \vec{a}_\rho\ cos\ \theta - \vec{a}_z\ sin\ \theta$ $\overrightarrow{a_\varphi} = \overrightarrow{a_\varphi}$
$\overrightarrow{a_x} = cos\ \varphi\ \overrightarrow{a_\rho} - sin\ \varphi\ \overrightarrow{a_\varphi}$ $\overrightarrow{a_y} = sin\ \varphi\ \overrightarrow{a_\rho} + cos\ \varphi\ \overrightarrow{a_\varphi}$ $\overrightarrow{a_z} = \overrightarrow{a_z}$	$\overrightarrow{a_x} = sin\ \theta\ cos\ \varphi\ \overrightarrow{a_r} + cos\ \theta\ cos\ \varphi\ \overrightarrow{a_\theta} - sin\ \varphi\ \overrightarrow{a_\varphi}$ $\overrightarrow{a_y} = sin\ \theta\ sin\ \varphi\ \overrightarrow{a_r} + cos\ \theta\ sin\ \varphi\ \overrightarrow{a_\theta} + cos\ \varphi\ \overrightarrow{a_\varphi}$ $\overrightarrow{a_z} = cos\ \theta\ \overrightarrow{a_r} - sin\ \theta\ \overrightarrow{a_\theta}$	$\vec{a}_\rho = \vec{a}_r\ sin\ \theta + \vec{a}_\theta\ cos\ \theta$ $\vec{a}_z = \vec{a}_r\ cos\ \theta - \vec{a}_\theta\ sin\ \theta$ $\overrightarrow{a_\varphi} = \overrightarrow{a_\varphi}$

<u>*Vector Expressions:*</u>

Also, for a vector $\vec{F}$ expressed in Cartesian, Cylindrical and Spherical coordinates as:

$$\vec{F} = F_x\ \vec{a}_x + F_y\ \vec{a}_y + F_z\ \vec{a}_z,$$
$$\vec{F} = F_\rho\ \vec{a}_\rho + F_\varphi\ \vec{a}_\varphi + F_z\ \vec{a}_z,$$
$$\vec{F} = F_r\ \vec{a}_r + F_\theta\ \vec{a}_\theta + F_\varphi\ \vec{a}_\varphi$$

Table 3.3 – Vector Expression Relations

Cylindrical to Cartesian	*Spherical to Cartesian*
$F_\rho = F_x\ cos\ \varphi + F_y\ sin\ \varphi$ $F_\varphi = -F_x\ sin\ \varphi + F_y\ cos\ \varphi$ $F_z = F_z$	$F_r = F_x\ sin\theta\ cos\varphi + F_y\ sin\theta\ sin\varphi + F_z\ cos\theta$ $F_\vartheta = F_x\ cos\theta\ cos\varphi + F_y\ cos\theta\ sin\varphi - F_z\ sin\theta$ $F_\varphi = -F_x\ sin\varphi + F_y\ cos\varphi$
$F_x = F_\rho\ cos\varphi - F_\varphi\ sin\ \varphi$ $F_y = -F_\rho\ sin\varphi + F_\varphi\ cos\ \varphi$ $F_z = F_z$	$F_x = F_r\ sin\theta\ cos\varphi + F_\theta\ cos\theta\ cos\varphi - F_\varphi\ sin\varphi$ $F_y = F_r\ sin\theta\ sin\varphi + F_\theta\ cos\theta\ sin\varphi + F_\varphi\ cos\varphi$ $F_z = F_r\ cos\theta - F_\theta\ sin\theta$

Addendum II

Vector Calculus

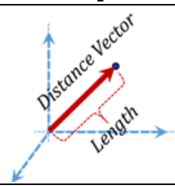
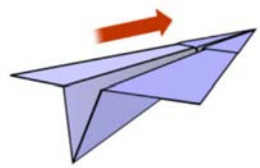
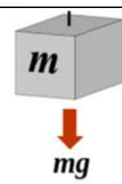
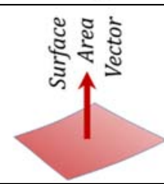
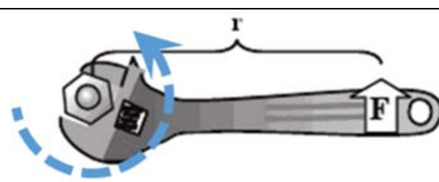
Vector Definition and Examples:

A vector is a quantity characterized by having a specific magnitude and a specific direction at each point in space. In this book will use the accented notation to denote vector, e.g., a vector "F" will appear as $\vec{F}$. A vector can be expressed as a scalar multiplied by a unit vector that has the same direction as the vector itself:

$$\vec{F} = F\,\vec{a}_F$$

Examples of vectors include distance, velocity, and force among many other physical quantities. The following table demonstrates some of these vector quantities.

Table 3.4 – Examples of Vectors

Vector	*Examples*
Length	
Velocity	
Weight	
Surface Area	
Force	
Torque	

Vector Representations in Coordinate Systems*:*

To express a vector in a coordinate system is to write the vector in terms of the three orthonormal components of that coordinate system. This is done by projecting the vector along the three directions of the coordinate system and finding the magnitude of each projection. If we denote the three projections of a vector $\vec{F}$ along the 3 coordinates of a "1-2-3" coordinate system as F1, F2, and F3, respectively, then we can write:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = F_1\,\vec{a}_1 + F_2\,\vec{a}_2 + F_3\,\vec{a}_3 \qquad (3.1)$$

where $\vec{a}_1$, $\vec{a}_1$, *and* $\vec{a}_1$ are the corresponding unit vectors of the coordinate system, Figure 3.21.

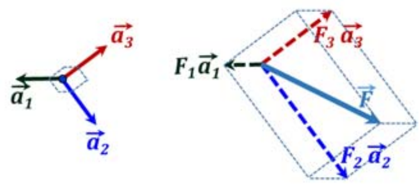


Figure 3.21

In the following, we will overview vector representation in the three coordinate systems. Two types of vectors will be demonstrated; distance vectors and others. The reason for this classification is that distance vectors are expressed in terms of the coordinate system dimensions and directions while the others will have different units but only expressed in terms of the coordinate system directions.

Vector Representation in a Cartesian coordinate system

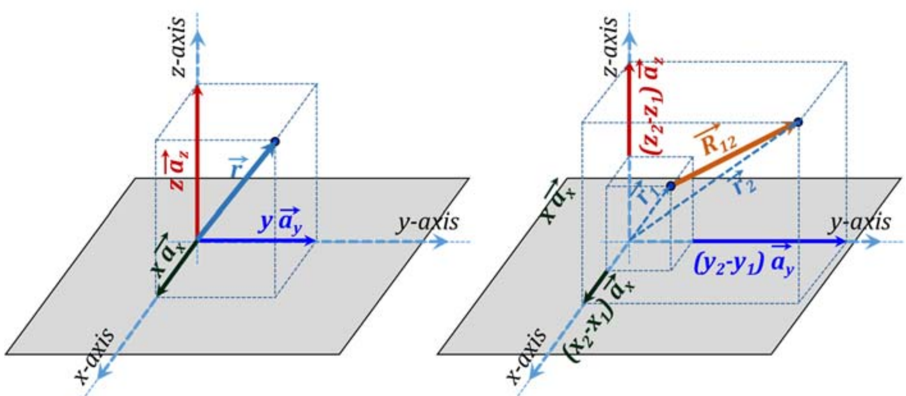


Figure 3.22

Figure 3.22 shows two types of distance vectors as represented in Cartesian coordinates; distance from the origin and distance between two points. The distance from the origin vector which we denote with the lower case r is in fact the same as the Spherical coordinate vector $\vec{r}$ which can be written as:

$$\vec{r} = x\,\overrightarrow{a_x} + y\,\overrightarrow{a_y} + z\,\overrightarrow{a_z} \qquad (3.2)$$

The distance between two points (1 and 2) will be denoted by the upper case vector $\vec{R}$. This can be expressed as the difference between two $\vec{r}$ vectors as follows:

$\overrightarrow{R_{12}} = \overrightarrow{r_2} - \overrightarrow{r_1} = (x_2\,\overrightarrow{a_x} + y_2\,\overrightarrow{a_y} + z_2\,\overrightarrow{a_z}) - (x_1\,\overrightarrow{a_x} + y_1\,\overrightarrow{a_y} + z_1\,\overrightarrow{a_z}) = (x_2 - x_1)\,\overrightarrow{a_x} + (y_2 - y_1)\,\overrightarrow{a_y} + (z_2 - z_1)\,\overrightarrow{a_z}$ *(3.3)*

Vector representation for a general form vector

Next, we view the vector representation for a general form vector $\vec{F}$ in Cartesian coordinates. This is demonstrated in Figure 3.23 yielding the analytical expression:

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z = F_x\,\vec{a}_x + F_y\,\vec{a}_y + F_z\,\vec{a}_z \qquad (3.4)$$

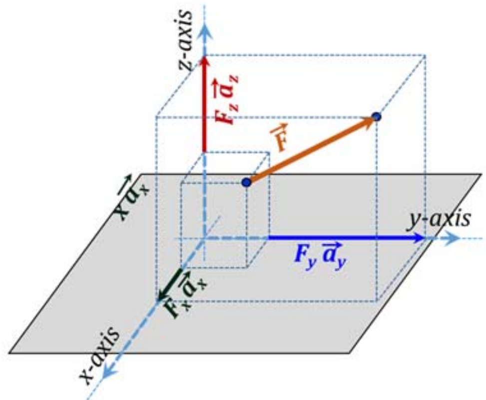


Figure 3.23

Vector Representation in Cylindrical coordinates

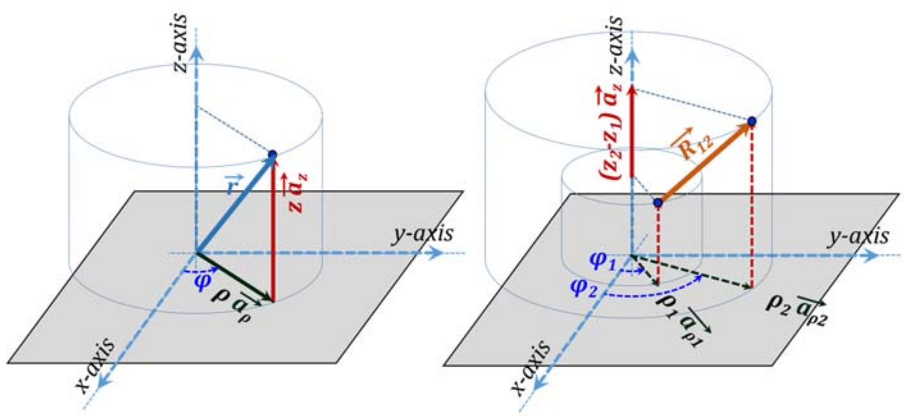


Figure 3.24

Figure 3.24 shows the two types of distance vectors as represented in Cylindrical coordinates. The distance from the origin vector $\vec{r}$ can be written as:

$$\vec{r} = \rho\,\overrightarrow{a_\rho} + z\,\overrightarrow{a_z} \qquad (3.5)$$

It is to be noted here that the $\overrightarrow{a_\rho}$ unit vector is a variable one which may cause some analytical challenges in some cases especially when integration is involved. When these challenges dominate, it may be "wiser" to switch to the Cartesian coordinate representation where all the unit vectors are constants.

The distance vector $\overrightarrow{R_{12}}$ between two points is also demonstrated in the figure. When expressed as the difference between two $\vec{r}$ vectors we be challenged by the vector subtraction of $\overrightarrow{a_{\rho 2}}$ *and* $\overrightarrow{a_{\rho 1}}$:

$$\overrightarrow{R_{12}} = \overrightarrow{r_2} - \overrightarrow{r_1} = \left(\rho_2\, \overrightarrow{a_{\rho 2}} + z_2\, \overrightarrow{a_z}\right) - \left(\rho_1\, \overrightarrow{a_{\rho 1}} + z_1\, \overrightarrow{a_z}\right) = \left(\rho_2\, \overrightarrow{a_{\rho 2}} - \rho_1\, \overrightarrow{a_{\rho 1}}\right) + (z_2 - z_1)\, \overrightarrow{a_z} = (x_2 - x_1)\, \overrightarrow{a_x} + (y_2 - y_1)\, \overrightarrow{a_y} + (z_2 - z_1)\, \overrightarrow{a_z} \qquad (3.6)$$

Again resorting to Cartesian coordinates offers a simple way to deal with this challenge.

Similarly, the general form vector $\vec{F}$ in Cylindrical coordinates is demonstrated in Figure 3.25. This corresponding analytical expression is:

$$\vec{F} = \overrightarrow{F_2} - \overrightarrow{F_1} = \left(F_{2\rho}\, \overrightarrow{a_{\rho 2}} + F_{2Z}\, \overrightarrow{a_z}\right) - \left(F_{1\rho}\, \overrightarrow{a_{\rho 1}} + F_{1Z}\, \overrightarrow{a_z}\right) = \left(F_{2\rho}\, \overrightarrow{a_{\rho 2}} - F_{1\rho}\, \overrightarrow{a_{\rho 1}}\right) + (F_{2Z} - F_{1Z})\, \overrightarrow{a_z} = F_x\, \overrightarrow{a_x} + F_y\, \overrightarrow{a_y} + F_z\, \overrightarrow{a_z} \qquad (3.7)$$

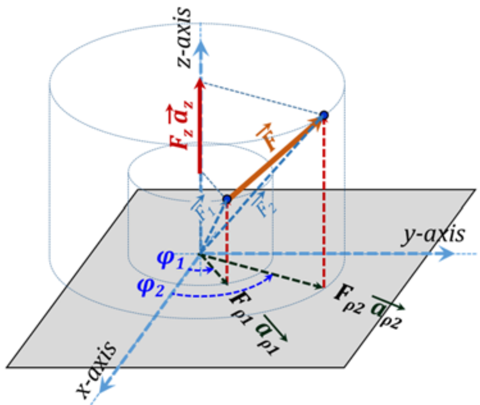


Figure 3.25

Vector Representation in Spherical coordinates

The case of Spherical coordinates, Figure 3.26, has challenges similar to those discussed in Cylindrical coordinate representation; all 3 unit vectors are variables. Switching to Cartesian offers the same convenience as discussed above.

$$\vec{F} = \overrightarrow{F_2} - \overrightarrow{F_1} = F_x\, \overrightarrow{a_x} + F_y\, \overrightarrow{a_y} + F_z\, \overrightarrow{a_z} \qquad (3.8)$$

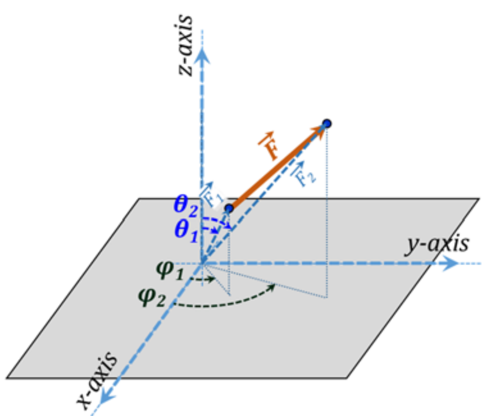


Figure 3.26

Vector Operations

The following table reviews various vector operations, gives examples of their graphical and physical applications, as well as their representations in the three coordinate systems. The "**red-bold**" terms in the expressions below indicate cases where variable vectors present mathematical challenges. In such cases, we may resort to the Cartesian representation where the unit vectors have constant directions at all times.

Table 3.5 – Vector Addition and Subtraction

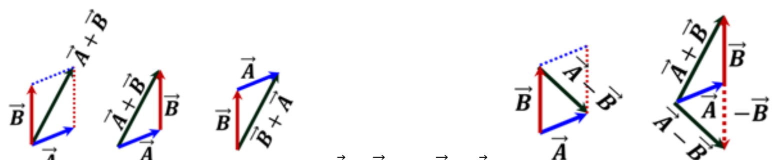
Addition & Subtraction	 $\vec{A} \pm \vec{B} = \pm\vec{B} + \vec{A}$
Cartesian	$\vec{A} \pm \vec{B} = A_x\,\overrightarrow{a_x} + A_y\,\overrightarrow{a_y} + A_z\,\overrightarrow{a_z} \pm (B_x\,\overrightarrow{a_x} + B_y\,\overrightarrow{a_y} + B_z\,\overrightarrow{a_z})$ $= (A_x \pm B_x)\,\overrightarrow{a_x} + (A_y \pm B_y)\,\overrightarrow{a_y} + (A_z \pm B_z)\,\overrightarrow{a_z}$
Cylindrical	$\vec{A} \pm \vec{B} = (A_\rho\,\overrightarrow{a_{\rho A}} + A_z\,\overrightarrow{a_z}) \pm (B_\rho\,\overrightarrow{a_{\rho B}} + B_z\,\overrightarrow{a_z})$ $= (\boldsymbol{A_\rho \overrightarrow{a_{\rho A}} \pm B_\rho \overrightarrow{a_{\rho B}}}) + (A_z \pm B_z)\overrightarrow{a_z}$ ➔ $= (A_x \pm B_x)\,\overrightarrow{a_x} + (A_y \pm B_y)\,\overrightarrow{a_y} + (A_z \pm B_z)\,\overrightarrow{a_z}$
Spherical	$\vec{A} \pm \vec{B} = (\boldsymbol{A_r \overrightarrow{a_{rA}} \pm B_r \overrightarrow{a_{rB}}})$ ➔ $= (A_x \pm B_x)\,\overrightarrow{a_x} + (A_y \pm B_y)\,\overrightarrow{a_y} + (A_z \pm B_z)\,\overrightarrow{a_z}$

Table 3.6 – Vector Scaling

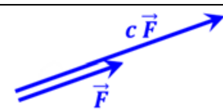
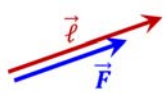
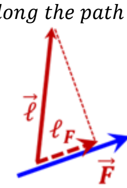
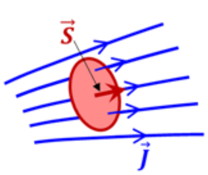
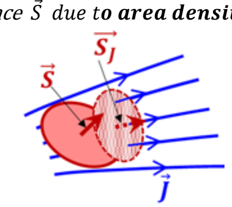
Scaling	
Cartesian	$\boldsymbol{c}\,(\boldsymbol{A_x\,\overrightarrow{a_x} + A_y\,\overrightarrow{a_y} + A_z\,\overrightarrow{a_z}}) = \boldsymbol{cA_x\,\overrightarrow{a_x} + cA_y\,\overrightarrow{a_y} + cA_z\,\overrightarrow{a_z}}$
Cylindrical	$\boldsymbol{c}\,(A_\rho\,\overrightarrow{a_\rho} + A_z\,\overrightarrow{a_z}) = \boldsymbol{cA_\rho\,\overrightarrow{a_\rho} + cA_z}\,\overrightarrow{a_z}$
Spherical	$c\,(A_r\,\overrightarrow{a_r}) = cA_r\,\overrightarrow{a_r}$

Table 3.7 – Vector Dot Product

Scalar (Dot) Product	*W = Work Done by force $\vec{F}$ along the path $\vec{\ell}$*  $W = \vec{F}.\vec{\ell} = F\,\ell$  $W = \vec{F}.\vec{\ell} = F\,\ell_F$
	*I = Current flow through the surface $\vec{S}$ due to **area density** $\vec{J}$*  $I = \vec{J}.\vec{S} = J\,S$  $I = \vec{J}.\vec{S} = J\,S_J$

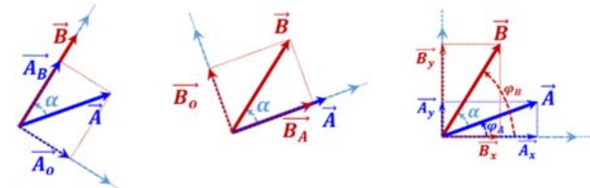
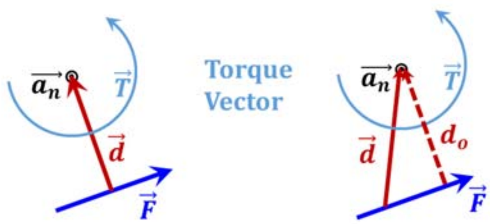
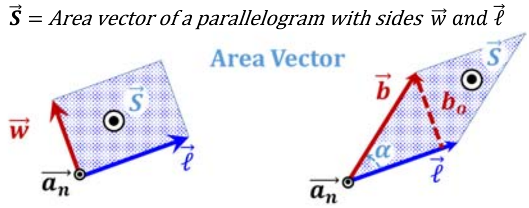
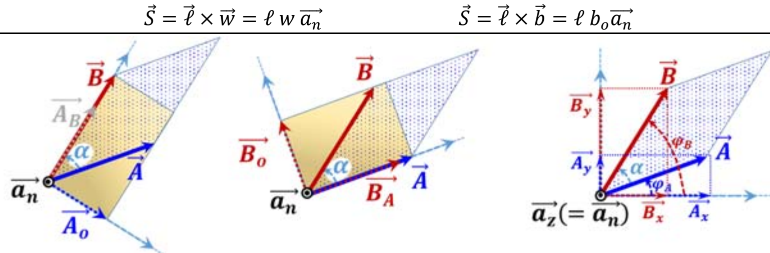
	 $\vec{A}\cdot\vec{B} = [A_B]\,B = [A\,cos(\alpha)]\,B = A\,B\,cos(\alpha)$ $\vec{A}\cdot\vec{B} = A\,[B_A] = A\,[B\,cos(\alpha)] = A\,B\,cos(\alpha)$ $\vec{A}\cdot\vec{B} = A_xB_x + A_yA_y = [A\,cos(\varphi_A)B\,cos(\varphi_B) + A\,sin(\varphi_A)B\,sin(\varphi_B)]$ $= A\,B\,[cos(\varphi_A)cos(\varphi_B) + sin(\varphi_A)sin(\varphi_B)] = AB[cos(\varphi_B - \varphi_A)]$ $= A\,B\,cos(\alpha)$
Cartesian	$\vec{A}\cdot\vec{B} = (A_x\,\overrightarrow{a_x} + A_y\,\overrightarrow{a_y} + A_z\,\overrightarrow{a_z})\cdot(B_x\,\overrightarrow{a_x} + B_y\,\overrightarrow{a_y} + B_z\,\overrightarrow{a_z}) = A_x\,B_x + A_y\,B_y + A_z\,B_z$
Cylindrical	$\vec{A}\cdot\vec{B} = (A_\rho\,\overrightarrow{a_{\rho A}} + A_z\,\overrightarrow{a_z})\cdot(B_\rho\,\overrightarrow{a_{\rho B}} + B_z\,\overrightarrow{a_z}) = A_\rho B_\rho(\overrightarrow{a_{\rho A}}\cdot\overrightarrow{a_{\rho B}}) + A_z\,B_z$
Spherical	$\vec{A}\cdot\vec{B} = (A_r\,\overrightarrow{a_{rA}})\cdot(B_r\,\overrightarrow{a_{rB}}) = A_rB_r(\overrightarrow{a_{rA}}\cdot\overrightarrow{a_{rB}})$

Table 3.8 – Vector Cross Product

Vector (Cross) Product	$\vec{T}$ = *Torque vector due to a force* $\vec{F}$ *and an arm* $\vec{d}$  $\vec{T} = \vec{F}\times\vec{d} = F\,d\,\overrightarrow{a_n}$ $\vec{T} = \vec{F}\times\vec{d} = F\,d_o\,\overrightarrow{a_n}$
	$\vec{S}$ = *Area vector of a parallelogram with sides* $\vec{w}$ *and* $\vec{\ell}$  $\vec{S} = \vec{\ell}\times\vec{w} = \ell\,w\,\overrightarrow{a_n}$ $\vec{S} = \vec{\ell}\times\vec{b} = \ell\,b_o\overrightarrow{a_n}$
	 $\vec{A}\times\vec{B} = [A_o]\,B\,\overrightarrow{a_n} = [A\,sin(\alpha)]\,B\,\overrightarrow{a_n} = A\,B\,sin(\alpha)\,\overrightarrow{a_n}$ $\vec{A}\times\vec{B} = A\,[B_o]\,\overrightarrow{a_n} = A\,[B\,sin(\alpha)]\,\overrightarrow{a_n} = A\,B\,sin(\alpha)\,\overrightarrow{a_n}$ $\vec{A}\times\vec{B} = [A_xB_y\,\overrightarrow{a_n} + A_yB_x(-\overrightarrow{a_n})] = [A\,cos(\varphi_A)Bsin(\varphi_B) - A\,sin(\varphi_A)B\,cos(\varphi_B)]\,\overrightarrow{a_n}$ $= A\,B\,[cos(\varphi_A)sin(\varphi_B) - sin(\varphi_A)cos(\varphi_B)]\,\overrightarrow{a_n}$ $= AB[sin(\varphi_B - \varphi_A)]\,\overrightarrow{a_n} = A\,B\,sin(\alpha)\,\overrightarrow{a_n}$
	$\vec{A}\times\vec{A} = 0$, $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
Cartesian	$\overrightarrow{a_x}\times\overrightarrow{a_y} = \overrightarrow{a_z}$, $\overrightarrow{a_y}\times\overrightarrow{a_z} = \overrightarrow{a_x}$, $\overrightarrow{a_z}\times\overrightarrow{a_x} = \overrightarrow{a_y}$

	$\vec{A} \times \vec{B} = (A_x \overrightarrow{a_x} + A_y \overrightarrow{a_y} + A_z \overrightarrow{a_z}) \times (B_x \overrightarrow{a_x} + B_y \overrightarrow{a_y} + B_z \overrightarrow{a_z}) = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$
Cylindrical	$\overrightarrow{a_\rho} \times \overrightarrow{a_\varphi} = \overrightarrow{a_z}$, $\overrightarrow{a_\varphi} \times \overrightarrow{a_z} = \overrightarrow{a_\rho}$, $\overrightarrow{a_z} \times \overrightarrow{a_\rho} = \overrightarrow{a_\varphi}$ $\vec{A} \times \vec{B} = (A_\rho \overrightarrow{a_{\rho A}} + A_z \overrightarrow{a_z}) \times (B_\rho \overrightarrow{a_{\rho B}} + B_z \overrightarrow{a_z})$ $= A_\rho B_\rho (\overrightarrow{\boldsymbol{a}_{\rho A}} \times \overrightarrow{\boldsymbol{a}_{\rho B}}) + (A_\rho B_z \overrightarrow{a_{\rho A}} \times \overrightarrow{a_z} + A_z B_\rho \overrightarrow{a_z} \times \overrightarrow{a_{\rho B}})$ $= A_\rho B_\rho (\overrightarrow{\boldsymbol{a}_{\rho A}} \times \overrightarrow{\boldsymbol{a}_{\rho B}}) + (-A_\rho B_z \overrightarrow{a_{\varphi A}} + A_z B_\rho \overrightarrow{a_{\varphi B}})$
Spherical	$\overrightarrow{a_r} \times \overrightarrow{a_\theta} = \overrightarrow{a_\varphi}$, $\overrightarrow{a_\theta} \times \overrightarrow{a_\varphi} = \overrightarrow{a_r}$, $\overrightarrow{a_\varphi} \times \overrightarrow{a_r} = \overrightarrow{a_\theta}$ $\vec{A} \times \vec{B} = (A_r \overrightarrow{a_{rA}}) \times (B_r \overrightarrow{a_{rB}}) = A_r B_r (\overrightarrow{\boldsymbol{a}_{rA}} \times \overrightarrow{\boldsymbol{a}_{rB}})$

Addendum III

Spatial Distributions and Densities

In the following we will discuss spatial distributions and densities of both static and dynamic quantities.

Static Distributions and Densities

Static quantities, such as mass, charge, or energy, may exist in several possible spatial distributions, Figure 3.27:

1. A concentration in an infinitesimally small volume, which we typically represent as a "point".
2. A distribution over a volume with infinitesimally small cross-section, which we typically represent as a "line".
3. A distribution over a volume with infinitesimally small thickness, which we typically represent as a "surface".
4. A distribution over a volume with non-zero dimensions.

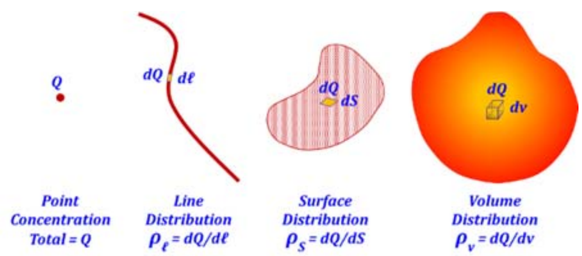


Figure 3.27

These distributions may or may not be uniform. To express their spatial features, we use appropriate density expressions such as volume, area, or linear density of the quantity distribution. Densities of static quantities are scalar in nature, i.e. have only magnitudes and do not have directions. We will use the subscripted ρ_v, ρ_S, and ρ_ℓ to denote volume, area, and linear densities, respectively. Hence, for the static quantity Q, we can write

$$\rho_v = \frac{dQ}{dv}\,,\quad \rho_S = \frac{dQ}{dS}\,,\quad and\ \rho_\ell = \frac{dQ}{d\ell} \qquad (3.9)$$

For uniform distributions, the density is constant at all the distribution locations, otherwise, the density would be a function of position. Depending on the case, these densities may or may not have physical relevance and their definition could be meaningless. Examples are defining a volumetric density for a point concentration where the volume is zero, or defining the linear density for a spherical volume distribution. The following table summarizes the corresponding densities for the four distribution forms.

Table 3.9 – Static Distribution Densities

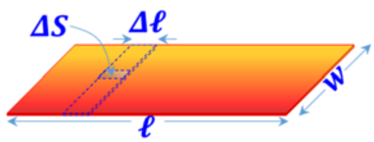
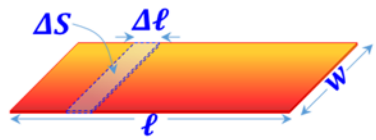
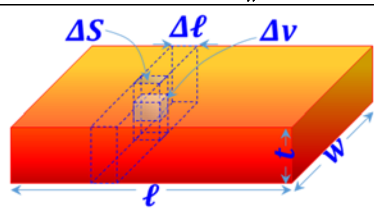
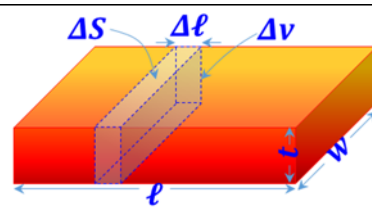
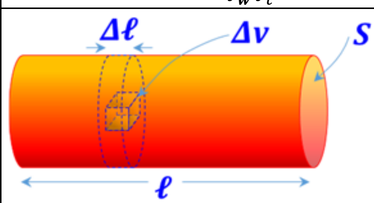
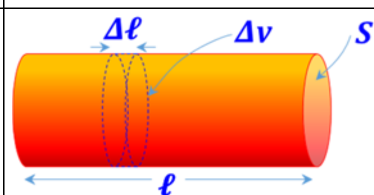
Configuration	*Volume Density= Quantity per unit volume*	*Surface Density= Quantity per unit area*	*Linear Density= Quantity per unit length*
Point Concentration	*Zero volume*➔ $\rho_v = \infty$	*Zero area*➔ $\rho_S = \infty$	*Zero length*➔ $\rho_\ell = \infty$
Line Distribution	*Zero volume*➔ $\rho_v = \infty$	*Zero area*➔ $\rho_S = \infty$	ρ_ℓ

Surface Distribution	*Zero volume➔* $\rho_v = \infty$	ρ_S	*(Atypical?)*
Volume Distribution	ρ_v	*(Atypical?)*	*(Atypical?)*

Conversions between static density expressions:

In some distribution configurations, more than one form of density can be simultaneously defined. An example is a cylindrical volume distribution for which a volumetric density can be defined while a linear density along the cylindrical axis can be defined as well. For such cases, it is useful to have conversion expressions between the different forms. In the following, we provide examples of such conversion relationships.

Table 3.10

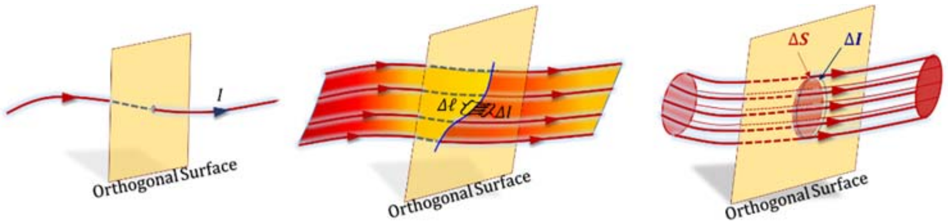
	Non uniform distribution	***Uniform distribution***
Sheet distribution $\rho_S = \frac{dQ}{dS}$ ***is defined***	 $\rho_\ell = \frac{\int_w \rho_S\, dS}{\Delta\ell} = \int_w \rho_S\, dw$	 $\rho_\ell = \rho_S\, w$
Rectangular prism (Slab) distribution $\rho_v = \frac{dQ}{dv}$ ***is defined***	 $\rho_S = \frac{\int_t \rho_v\, dv}{\Delta S} = \int_t \rho_S\, dt$ $\rho_\ell = \frac{\int_w \int_t \rho_v\, dv}{\Delta\ell} = \int_w \int_t \rho_v\, dt\, dw$	 $\rho_S = \rho_S\, t$ $\rho_\ell = \rho_v\, t\, w$
Circular prism (Cylinder) distribution $\rho_v = \frac{dQ}{dv}$ ***is defined***	 $\rho_\ell = \frac{\int_S \rho_v\, dv}{\Delta\ell} = \int_S \rho_v\, dS$	 $\rho_\ell = \rho_v\, S$

Dynamic Distributions and Densities:

Examples of dynamic quantities include electric current, air current, fluid flow, and energy flow. Spatial distributions for a dynamic flow can only exist in one of the following forms, Figure 3.28:

1. A stream distribution with infinitesimally small cross-section, which we typically represent as a "line current/flow".

2. A stream distribution with infinitesimally small thickness, which we typically represent as a “surface current / laminar flow”.
3. A stream distribution with non-zero cross-sectional area, “volume current/flow”.



Line Current
Total Current I

Surface Current
Linear Density of Surface Current $K = \frac{\Delta I}{\Delta \ell}$

Volume Current
Surface Density of Volume Current $J = \frac{\Delta I}{\Delta S}$

Figure 3.28

These distributions may or may not be uniform. To express their spatial features, in both magnitude and direction, we use appropriate vector density expressions.

$$\vec{J} = \frac{dI}{dS_n}\vec{a}_I \quad and \quad \vec{K} = \frac{dI}{d\ell_n}\vec{a}_I \qquad (3.10)$$

where the n subscripts indicate that both $d\ell$ and dS must be orthogonal to the flow and $\vec{a}_I$ is the unit vector along the current/flow direction. For uniform distributions, the densities are constant at all the distribution locations, otherwise, the density would be a function of position. Again, depending on the case, these densities may or may not have physical relevance and their definition could be meaningless. The following table summarizes the corresponding densities for dynamic distributions.

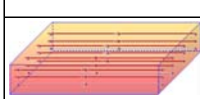
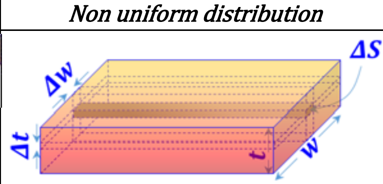
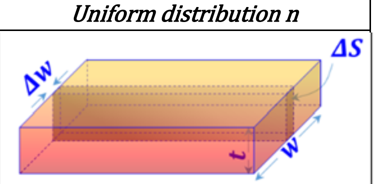
Table 3.11 – Dynamic Distribution Densities

Configuration	*Area Density= Quantity per unit area*	*Linear Density= Quantity per unit length*
Line Stream Concentration	*Zero area*➔ $\vec{J} = \infty$	*Zero length*➔ $\vec{K} = \infty$
Surface Stream Distribution	*Zero area*➔ $\vec{J} = \infty$	$\vec{K}$
Volume Stream Distribution	$\vec{J}$	*(Atypical?)*

Conversions between dynamic density expressions:

In this section, we demonstrate the relationship between the two forms of dynamic distribution densities in cases where both can be simultaneously defined. In the table below, we the example of a stream distribution in a slab configuration and the corresponding conversion relationships.

Table 3.12

	Non uniform distribution	***Uniform distribution n***
 ***Slab Stream Flow/Current*** $\vec{J} = \frac{dI}{dS_n}\vec{a}_I$ ***is defined***	 $\vec{K} = \frac{\int_t J\, dS}{\Delta w}\,\vec{a}_I = \int_t J\, dt\ \vec{a}_I$	 $\vec{K} = J\, t\, \vec{a}_I$

Addendum IV

Line, Surface, and Volume Integrations

Introduction

In the course of this book, as we deal with various physical phenomena, analyses involving integration of scalar and vector quantities is common. We often need to carry out contour (or line) integrations, integrations over an area of a surface, a closed surface as well as volume integrations. Our background in mathematics should enable us to carry most of these integrations once they are set up properly. We can also resort to integration tables, computer software packages, or even numerical tools for "difficult" integrations. Hence, in this addendum, we will focus on two aspects of the issue that are often the obstacle. One is how to set up the integration equation starting with the physical problem, and the second is how to deal with vector quantities within the integrand.

The first obstacle of setting up the integral is an "integral" part of setting up the proper mathematical model of the physical problem. This is an acquired skill that the learners in this field acquire with practice. The learner needs to get exposed to a variety of cases and a variety of analysis tools to appreciate what works and what does not and when to use a specific model and what are the constraints of that use. Gaining this skill requires proper appreciation to the physics of the subject and good command of relevant mathematical tools. This will be demonstrated and emphasized throughout the different chapters of this book.

We now turn to dealing with integrations containing vector quantities. This will be followed by an overview/survey of line, surface, and volume "scalar" integrations.

Integrating vector quantities

To integrate vector quantities is simply performing vector summation of incremental vector elements. Since the sum of vectors is controlled by the directions of the vectors involved, the process of vector integration must take into account the variability of the direction of the vectors being integrated. Let us start with a "sarcastic" but true statement by saying that "The proper way of handling vector integration is not to do vector integration." What is meant here is that we should not start the integration process before we remove all vector quantities from within the integrand.

The process involves finding ways to get all vector terms outside the integration sign leaving only scalar quantities inside the integration. To extract a term outside the integral operator, this term must be a constant. Hence, what we must do is express all variable-direction vectors in terms of fixed-direction vectors that can then be extracted outside the integration. Let us cite some examples to demonstrate.

Example of a Cartesian coordinate Integral

$$\int \vec{x}\, dx = \int x\, \vec{a}_x\, dx = \vec{a}_x \int x\ \ dx = \vec{a}_x \left(\frac{x^2}{2} + c\right) \qquad (3.11)$$

$$\int \vec{y}\, dx = \int y\, \vec{a}_y\, dx = \vec{a}_y \int y\ \ dx \qquad (3.12)$$

Example of a Cylindrical coordinate Integral

$$\int \vec{\rho}\, d\rho = \int \rho\, \vec{a}_\rho\, d\rho = \vec{a}_\rho \int \rho\ \ d\rho = \vec{a}_\rho \left(\frac{\rho^2}{2} + c\right) \qquad (3.13)$$

For an integration along $d\rho$, the "$\vec{a}_\rho$ " has a constant direction and hence it is a constant vector that can be taken outside the integration. However if the same integration was along $d\varphi$, the "$\vec{a}_\rho$" will vary as we vary φ and hence "$\vec{a}_\rho$" needs to be expressed in terms of other constant-direction vectors to be able to carry out the integration. Logically, we use the Cartesian unit vectors "$\vec{a}_x, \vec{a}_y, and\ \vec{a}_z$" which are always constant vectors in this regard,

$$\int \vec{\rho}\, d\varphi = \int \rho\, \vec{a}_\rho\, d\varphi = \int \rho \left(\cos\varphi\, \overrightarrow{a_x} + \sin\varphi\, \overrightarrow{a_y}\right) d\varphi = \int \rho \left(\cos\varphi\, \overrightarrow{a_x}\right) d\varphi + \int \rho \left(\sin\varphi\, \overrightarrow{a_y}\right) d\varphi = \overrightarrow{a_x} \int \rho\, (\cos\varphi)\, d\varphi + \overrightarrow{a_y} \int \rho (\sin\varphi)\, d\varphi = \overrightarrow{a_x}\, (\rho \sin\varphi + c_1) - \overrightarrow{a_y}\, \rho\, (\cos\varphi + c_2) \quad (3.14)$$

Adding the limits between 0 and 2π, this integration reduces to:

$$\int_0^{2\pi} \vec{\rho}\, d\varphi = \overrightarrow{a_x}\ \rho \int_0^{2\pi} (\cos\varphi)\, d\varphi + \overrightarrow{a_y}\, \rho \int_0^{2\pi} (\sin\varphi)\, d\varphi = 0 \quad (3.15)$$

This result makes physical sense since adding the variable direction vector $\vec{\rho}$ around a complete 2π (360 degrees) rotation will produce zero net, see Figure 3.29.

Figure 3.29

Example of a Spherical coordinate Integral

$$\int \vec{r}\, dr = \int r\, \vec{a}_r\, dr = \vec{a}_r \int r\, dr = \vec{a}_r\, (r^2/2) \quad (3.16)$$

$$\int \vec{r}\, d\theta = \int r\, \vec{a}_r\, d\theta = \int r \left(\sin\theta \cos\varphi\, \overrightarrow{a_x} + \sin\theta \sin\varphi\, \overrightarrow{a_y} + \cos\theta\, \overrightarrow{a_z}\right) d\theta = \overrightarrow{a_x} \int r \sin\theta \cos\varphi\, d\theta + \overrightarrow{a_y} \int r \sin\theta \sin\varphi\, d\theta + \overrightarrow{a_z} \int r \cos\theta\, d\theta \quad (3.17)$$

Integrating scalar quantities

In this section, we will survey a few examples of line, surface and volume integrations that we will find relevant in the chapters ahead.

Examples of Density Integrations:

Two types of densities are reviewed here; static and dynamic. Examples of static densities include mass and charge distributions, while examples of dynamic densities include fluid flow and electric current.

Static Linear Density

If $\rho_\ell(position)$ is the linear density of the distribution of a quantity Q, we can obtain the total Q by integrating the linear density over the length of the distribution.

$$Q = \int dQ = \int \frac{dQ}{d\ell}\, d\ell = \int \rho_\ell\, d\ell \quad (3.18)$$

Static Surface Density

If $\rho_S(position)$ is the surface density of the distribution of a quantity Q, we can obtain the total Q by integrating the surface density over the area of the distribution.

$$Q = \int dQ = \iint \frac{dQ}{dS}\, dS = \iint \rho_S\, dS \quad (3.19)$$

Static Volumetric Density

If $\rho_v(position)$ is the volumetric density of the distribution of a quantity Q, we can obtain the total Q by integrating the volumetric density over the volume of the distribution.

$$Q = \int dQ = \iiint \frac{dQ}{dv}\, dv = \iiint \rho_v\, dv \qquad (3.20)$$

Dynamic Linear Density

If $K_\ell(position)$ is the linear density of the distribution of a current (flow) I, we can obtain the total I by integrating the linear density across the cross-sectional length of the distribution. The cross-sectional length is a length orthogonal to the flow distribution, Figure 3.30.

$$I = \int dI = \int \frac{dI}{d\ell}\, d\ell = \int K_\ell\, d\ell \qquad (3.21)$$

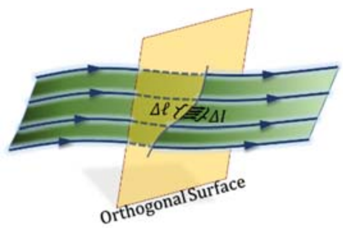


Figure 3.30

Dynamic Surface Density

The total flow (current), I, flowing through a specific surface is the integration of the flow surface density (J=dI/dS) over the cross-sectional surface area of interest. The cross-sectional area is that of a surface orthogonal to the flow distribution, Figure 3.31.

$$I = \int dI = \iint \frac{dI}{ds}\, dS = \iint J\, dS \qquad (3.22)$$

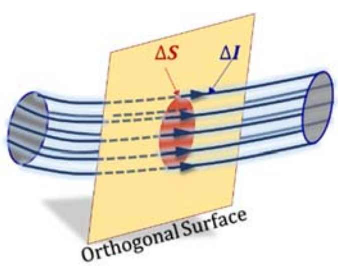


Figure 3.31

Examples of Work and Energy Integrations:

The work done by a force acting on an object as it moves an unconstrained object a certain distance (along the action line of the force), see Figure 3.32.

$$W = \int dW = \int F\, d\ell \qquad (3.23)$$

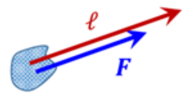


Figure 3.32

The total energy stored in a specific volume is the integration of the volumetric energy density *(w=dU/dv)* over the volume of interest.

$$U = \int dU = \iiint \frac{dU}{dv}\, dv = \iiint w\, dv \qquad (3.24)$$

Table 3.13 - Examples of integrations in different coordinate systems

	Line: $\int F\, d\ell$	**Surface:** $\iint J\, dS$	**Volume:** $\iiint w\, dv$
Cartesian	$\int F(x,y,z)\, dx$	$\iint J(x,y,z)\, dydz$	$\iiint w(x,y,z)\, dx\, dy\, dz$
	$\int F(x,y,z)\, dy$	$\iint J(x,y,z)\, dzdx$	
	$\int F(x,y,z)\, dz$	$\iint J(x,y,z)\, dxdy$	
Cylindrical	$\int F(\rho,\varphi,z)\, d\rho$	$\iint J(\rho,\varphi,z)\, \rho d\varphi\, dz$	$\iiint w(\rho,\varphi,z)\, d\rho\, \rho d\varphi\, dz$
	$\int F(\rho,\varphi,z)\, \rho\, d\varphi$	$\iint J(\rho,\varphi,z)\, dz\, d\rho$	
	$\int F(\rho,\varphi,z)\, dz$	$\iint J(\rho,\varphi,z)\, d\rho\, \rho d\varphi$	
Spherical	$\int F(r,\theta,\varphi)\, dr$	$\iint J(r,\theta,\varphi)\, rd\theta\, rsin\theta\, d\varphi$	$\iiint w(r,\theta,\varphi)\, dr\, rd\theta\, rsin\theta\, d\varphi$
	$\int F(r,\theta,\varphi)\, r\, d\theta$	$\iint J(r,\theta,\varphi)\, dr\, rsin\theta\, d\varphi$	
	$\int F(r,\theta,\varphi)\, r\, sin\theta\, d\varphi$	$\iint J(r,\theta,\varphi)\, dr\, rd\theta$	